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ON TEICHMÜLLER'S THEOREM ON THE QUASI-INVARIANCE OF CROSS RATIOS[†]

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ABSTRACT

Teichmüller's theorem gives necessary and sufficient conditions for mapping one ordered quadruple by a K-quasiconformal map onto a second ordered quadruple. We give a simple non-computational proof of the necessity part. We then characterize such extremal mappings, and obtain as a consequence a new formula for the modular function, which leads to a very simple derivation of the known expression for the Poincaré metric on the thrice-punctured sphere.

§1. Let (a_1, a_2, a_3, a_4) and $(a_1^*, a_2^*, a_3^*, a_4^*)$ be two ordered quadruples of distinct points in the extended complex plane, $\mathbf{C} \cup \{\infty\}$. Form the cross ratios

$$\alpha = \frac{a_3 - a_1}{a_2 - a_1} \div \frac{a_3 - a_4}{a_2 - a_4}, \qquad \alpha^* = \frac{a_3^* - a_1^*}{a_2^* - a_1^*} \div \frac{a_3^* - a_4^*}{a_2^* - a_4^*}.$$

(Of course, α is the image of a_1 under the Möbius transformation that sends a_3, a_4, a_2 to 0, 1, ∞.)

The cross ratios are points in the twice-punctured plane $\mathbf{C} \setminus \{0,1\}$. Let $\sigma(\cdot,\cdot)$ denote the non-Euclidean Poincaré distance on $\mathbf{C} \setminus \{0,1\}$. It is obtained by projecting the Poincaré metric on the unit disc Δ

$$\frac{1}{1-|z|^2}|dz|$$

to $\mathbf{C} \setminus \{0,1\}$. If the corresponding distance is denoted by $\rho(\cdot,\cdot)$, then by choosing a holomorphic universal covering map

$$\pi : \Delta \to \mathbf{C} \setminus \{0,1\},$$

we have

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$$\sigma(a,b)=\inf_{\substack{z\in\pi^{-1}(a)\\ \zeta\in\pi^{-1}(b)}}\rho(z,\zeta).$$

Further, if we fix any $z_0\in\pi^{-1}(a)$, it is always possible to pick a $\zeta_0\in\pi^{-1}(b)$ such that

$$\sigma(a,b)=\rho(z_0,\zeta_0).$$

It shall occasionally be more convenient to use the upper half plane U as the model for the universal covering of $\mathbf{C}\setminus\{0,1\}$ (recall that the Poincaré metric on U is given by $|dz|/|z-\bar{z}|$). We shall however use π and ρ to represent also the covering of $\mathbf{C}\setminus\{0,1\}$ by U and the distance in U, respectively. Similarly, we let Γ denote the covering group of π. This group can be described quite explicitly. We shall not need this description at all.

In the course of another investigation (that will be reported elsewhere), I came across a short proof of the "only if" part of the following important results of Teichmüller.

THEOREM 1 (Teichmüller [7], Ahlfors [2]). *There exists a K-quasiconformal automorphism of $\mathbf{C}\cup\{\infty\}$ which maps the ordered quadruple (a_1,a_2,a_3,a_4) onto the ordered quadruple $(a_1^*,a_2^*,a_3^*,a_4^*)$ if and only if $\sigma(\alpha,\alpha^*)\leqq\frac{1}{2}\log K$.*

PROOF. For subsequent use we reproduce a proof of the "if" part (see Ahlfors [3], Chapter III). Let A and A^* be Möbius transformations that send the ordered quadruples (a_1,a_2,a_3,a_4) and $(a_1^*,a_2^*,a_3^*,a_4^*)$ into the ordered quadruples $(0,1,\alpha,\infty)$ and $(0,1,\alpha^*,\infty)$, respectively (invariance of cross ratios under Möbius transformations). Let us choose points, τ, $\tau^*\in U$ such that

$$\rho(\tau,\tau^*)=\sigma(\alpha,\alpha^*)=\tfrac{1}{2}\log K,$$

and

$$\pi(\tau)=\alpha,\quad \pi(\tau^*)=\alpha^*.$$

It suffices to produce a K-quasiconformal mapping of $\mathbf{C}_\alpha=\mathbf{C}\setminus\{0,1,\alpha\}$ onto $\mathbf{C}_{\alpha^*}=\mathbf{C}\setminus\{0,1,\alpha^*\}$ that preserves the ordered triples.

Consider the complex plane $\mathbf{C}$ factored by the group of motions G_τ generated by

$$z\mapsto z+1,\quad z\mapsto z+\tau,\quad z\mapsto -z.$$

The factor space $\mathbf{C}/G_\tau$ is, of course, conformally equivalent to the sphere $\mathbf{C}\cup\{\infty\}$. Observe that G_τ leaves invariant $\mathbf{P}_\tau$, the plane punctured at the lattice points

$$\frac{n}{2}+\frac{m}{2}\tau,\ n\in\mathbf{Z},\ m\in\mathbf{Z},$$

and that $\mathbf{P}_\tau/G_\tau$ is a sphere punctured at four points. Now the Weierstrass $\wp$-function

$$\wp(\tau,z)=\frac{1}{z^2}+\sum_{\substack{(n,m)\in Z^2\\(n,m)\neq(0,0)}}\left[\frac{1}{(z-n-m\tau)^2}-\frac{1}{(n+m\tau)^2}\right]$$

is invariant under G_τ and maps $\mathbf{P}_\tau$ onto a sphere punctured at four points. It is necessary to evaluate the punctures: $\wp(0)$, $\wp(\frac{1}{2})$, $\wp(\tau/2)$, $\wp((1+\tau)/2)$. Clearly $\wp(0)=\infty$. Set

$$e_1=\wp\left(\frac{1}{2}\right),\quad e_2=\wp\left(\frac{\tau}{2}\right),\quad e_3=\wp\left(\frac{1+\tau}{2}\right).$$

It is known that the modular function on U defined by

$$\lambda(\tau)=\frac{e_3-e_1}{e_2-e_1}$$

is a holomorphic universal covering map of $\mathbf{C}\setminus\{0,1\}$, and thus we may assume that $\lambda=\pi$. Thus to show that $\mathbf{C}_\alpha=\mathbf{C}\setminus\{e_1,e_2,e_3\}$, it suffices to prove that the ordered quadruple $\{e_1,e_2,e_3,\infty\}$ has cross ratio α. But this is precisely the meaning of the equation $\lambda(\tau)=\pi(\tau)=\alpha$.

We construct a K-quasiconformal automorphism of $\mathbf{C}$ by

$$f(z)=\frac{(\tau^*-\bar{\tau})z+(\tau-\tau^*)\bar{z}}{\tau-\bar{\tau}},\qquad z\in\mathbf{C}. \tag{1}$$

Note that f has Beltrami coefficient $(=f_{\bar z}/f_z)$ $(\tau-\tau^*)/(\tau^*-\bar\tau)$ (of absolute value <1) and hence dilatation K. Further f fixes $\mathbf{R}$ and

$$f(-z)=-f(z),\ f\left(z+\frac{1}{2}\right)=f(z)+\frac{1}{2},\ f\left(z+\frac{\tau}{2}\right)=f(z)+\frac{\tau^*}{2}.$$

These relations imply first that f maps $\mathbf{P}_\tau$ onto $\mathbf{P}_\tau^*$ and that f conjugates G_τ into G_{τ^*}, and thus projects to a K-quasiconformal mapping of $\mathbf{C}_\alpha$ onto $\mathbf{C}_{\alpha^*}$.

We are now going to prove the "only if" part. Consider the Banach space M of Beltrami coefficients; that is, M is the open unit ball in $L^\infty(\mathbf{C})$. Let $\|\cdot\|$ denote the norm in this Banach space. A basic theorem of Ahlfors–Bers [4] asserts that for each $\mu\in M$, there exists a unique quasiconformal automorphism w^μ of $\mathbf{C}\cup\{\infty\}$ with Beltrami coefficient μ that fixes the points $0, 1, \infty$. Further for fixed $z\in\mathbf{C}$,

$$M \ni \mu \mapsto w^{\mu}(z) \in \mathbf{C}$$

is a holomorphic funtion. Now let $K(w^{\mu})$ be the dilatation of the quasiconformal automorphism w^{μ} $(K(w^{\mu}) = (1 + \|\mu\|)/(1 - \|\mu\|))$. The Banach modelled manifold M has a Kobayashi metric d. It is quite elementary to see that (because M is a ball)

$$d(\mu, \nu) = \tfrac{1}{2}\log K(w^{\mu} \circ (w^{\nu})^{-1}).$$

Now, the Kobayashi metric on $\mathbf{C}\setminus\{0,1\}$ agrees with the Poincaré metric (Kobayashi [6], Chapter IV). Hence any holomorphic mapping

$$\varphi : M \to \mathbf{C}\setminus\{0,1\}$$

is distance decreasing with respect to d and σ. Let (a, b, c) be three distinct points in $\mathbf{C}$. Then

$$\varphi_{abc}(\mu) = \frac{w^{\mu}(c) - w^{\mu}(a)}{w^{\mu}(b) - w^{\mu}(a)}$$

defines a holomorphic map of M into $\mathbf{C}\setminus\{0,1\}$. Thus

$$\sigma\left(\varphi_{abc}(\mu), \frac{c-a}{b-a}\right) \leqq d(\mu, 0) \leqq \frac{1}{2}\log K$$

provided that $K(w^{\mu}) \leqq K$. Taking the distinct points $(0, 1, z)$, we see that

$$\sigma(w^{\mu}(z), z) \leqq \tfrac{1}{2}\log K, \qquad \text{for } K(w^{\mu}) \leqq K,$$

which is enough to complete the proof of the theorem.

REMARKS. (1) We did not need to know the Kobayashi metric on M to obtain the above result. It sufficed to use the Poincaré metric on the discs in M of the form

$$\left\{ z\frac{\mu}{\|\mu\|};\ |z| < 1, \mu \neq 0 \right\}.$$

(2) Agard [1] also proved the "only if" part of the above theorem by relying on an accurate formula for σ. Ahlfors' proof [3, Chapter III] is also more computational than the one given above. However, the Ahlfors proof is more elementary since it does not rely on the generalized Riemann mapping theorem (Ahlfors–Bers [4]).

§2. A quasiconformal automorphism w_0 mapping the ordered quadruple of distinct points (a_1, a_2, a_3, a_4) onto the ordered quadruple $(a_1^*, a_2^*, a_3^*, a_4^*)$ is called *extremal* if $K(w_0) \leqq K(w)$ for all w with the same property.

THEOREM 2. *There exists an extremal mapping f taking the ordered quadruple (a_1, a_2, a_3, a_4) onto the ordered quadruple $(a_1^*, a_2^*, a_3^*, a_4^*)$. This mapping has dilatation $K = \exp 2\sigma(\alpha, \alpha^*)$, and Beltrami coefficient $\mu = k\bar{\varphi}/|\varphi|$, where $k = (K-1)/(K+1)$, and $\varphi(z)dz^2$ is a holomorphic quadratic differential on $\mathbf{C} \cup \{\infty\}$ (thus φ is a rational function) with simple poles at a_1, a_2, a_3, a_4. In particular, all the extremal mappings are Teichmüller mappings. The extremal mapping is unique if and only if for each $\tau \in U$ with $\pi(\tau) = \alpha$ there is a unique $\tau^* \in U$ with $\pi(\tau^*) = \alpha^*$ and $\rho(\tau, \tau^*) = \sigma(\alpha, \alpha^*)$. In particular, there exists an $r > 0$ such that every Teichmüller mapping w with Beltrami coefficient $t\bar{\varphi}/|\varphi|$ with $|t| < r$ is the unique extremal taking $\mathbf{C} \cup \{\infty\} \setminus \{a_1, a_2, a_3, a_4\}$ onto $\mathbf{C} \cup \{\infty\} \setminus \{w(a_1), w(a_2), w(a_3), w(a_4)\}$.*

PROOF. We have shown the existence of an extremal f with $K(f) = \exp 2\sigma(\alpha, \alpha^*)$. The function f constructed by formula (1) is a Teichmüller mapping since the constant function is a quadratic differential for G_τ. Now assume that w is an extremal function, then $K(w) = \exp 2\sigma(\alpha, \alpha^*)$. Let us replace w by w_1 which is a Teichmüller mapping in the same homotopy class as $w: \mathbf{C} \cup \{\infty\} \setminus \{a_1, a_2, a_3, a_4\} \to \mathbf{C} \cup \{\infty\} \setminus \{w(a_1), w(a_2), w(a_3), w(a_4)\}$. Since w is homotopic to w_1, $w(a_j) = w_1(a_j)$ for $j = 1, 2, 3, 4$. Since w_1 is a Teichmüller mapping $K(w_1) \leqq K(w)$. Since $\sigma(\alpha, \alpha^*) = \frac{1}{2}\log K(w)$, $K(w_1) \geqq K(w)$. By uniqueness of Teichmüller mappings (see, for example, Bers [5]) $w = w_1$. The dsicussion of §1 established the necessary and sufficient conditions for uniqueness of extremal mappings.

Now let ω be a fundamental domain for Γ centered at the origin (we are assuming Γ acts on Δ). Let r be chosen in such a way that the closed non-Euclidean ball of radius r is contained in the interior of ω. Thus r satisfies the conditions of the theorem.

AN IMPORTANT PROBLEM. Let Γ be a Fuchsian group operating on U. Let $a, b, c \in \mathbf{R}$ with $a < b < c$. Let μ be an extremal Beltrami coefficient for Γ with support in U. Then w^μ is, of course, conformal in U^*, the lower half plane. In order to determine the Carethéodory metric on the Teichmüller space of Γ, it is important to evaluate

$$\sup_{\substack{a, b, c \in \mathbf{R} \\ a < b < c}} \sigma\left(\frac{w^\mu(b) - w^\mu(c)}{w^\mu(a) - w^\mu(c)}, \frac{b-c}{a-c}\right).$$

Estimates (lower bounds) for the above expression in terms of $K(w^\mu)$ would be helpful. Theorem 2 showed that the above supremum is less than $\frac{1}{2}\log K(w^\mu)$ even in the one-dimensional Teichmüller spaces.

§3. Let us now fix our attention on the quadruple $(0, 1, \alpha, \infty)$ with $\alpha \in \mathbf{C}\setminus\{0, 1\}$. Then

$$\frac{1}{\zeta(\zeta-1)(\zeta-\alpha)}d\zeta^2 = \varphi(\zeta)d\zeta^2$$

is a quadratic differential on $\mathbf{C}\cup\{\infty\}$ with simple poles at the four points: $0, 1, \alpha, \infty$. Let $\mu = \bar{\varphi}/|\varphi|$. Thus μ is, of course, a Teichmüller differential of norm 1. For $z \in \mathbf{C}$, $|z| < 1$, we can form the normalized $z\mu$-conformal automorphism of $\mathbf{C}\cup\{\infty\}$, $w^{z\mu}$. We have shown that for $|z|$ small

$$\sigma(w^{z\mu}(\alpha), \alpha) = \frac{1}{2}\log\frac{1+|z|}{1-|z|} = \rho(z, 0).$$

Let us now fix α and define $f(z) = w^{z\mu}(\alpha)$. Then

$$f\colon \Delta \to \mathbf{C}\setminus\{0, 1\}$$

is a holomorphic function. Since Δ is simply connected, by the Monodromy theorem, there exists a holomorphic function $h: \Delta \to \Delta$ such that $\pi \circ h = f$. Now each of these three functions does not increase the Poincaré distance. Since $\sigma(f(0), f(z)) = \rho(0, z)$ for $|z|$ small, we conclude that for small $|z|$, we have $\rho(h(0), h(z))) = \rho(0, z)$. Hence by Schwarz's lemma h is a rotation. We have extablished the following result.

THEOREM 3. *The function $z \mapsto w^{z\mu}(\alpha)$ is a holomorphic universal covering map of $\mathbf{C}\setminus\{0, 1\}$ by Δ that takes 0 into α.*

COROLLARY (Agard [1]). *The (infinitesimal) Poincaré metric on $\mathbf{C}\setminus\{0, 1\}$ is given by*

$$g(\alpha)|d\alpha|,$$

where

$$g(\alpha) = \left(\frac{|\alpha||\alpha-1|}{2\pi}\int\int_{\mathbf{C}}\frac{|d\zeta \wedge d\bar{\zeta}|}{|\zeta||\zeta-1||\zeta-\alpha|}\right)^{-1}.$$

PROOF. The Poincaré metric on $\mathbf{C}\setminus\{0, 1\}$ is defined by the invariant expression

$$\frac{1}{1-|z|^2}|dz| = g(f(z))|df(z)|,$$

where $f: \Delta \to \mathbf{C} \setminus \{0, 1\}$ is any covering map. Thus for our f and $z = 0$,

$$g(\alpha) = |f'(0)|^{-1}.$$

But (see Ahlfors–Bers [4])

$$f'(0) = \frac{\partial w^{z\mu}(\alpha)}{\partial z}\bigg|_{z=0} = \frac{\alpha(\alpha-1)}{2\pi i}\int\int_{\mathbf{C}} \frac{\mu(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta(\zeta-1)(\zeta-\alpha)}$$

$$= \frac{\alpha(\alpha-1)}{2\pi i}\int\int_{\mathbf{C}} |\varphi(\zeta)| d\zeta \wedge d\bar{\zeta}.$$

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